The Asymptotic Cone of the Heisenberg Group

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This is from Enrico Le Donne's book *Metric Lie Groups*. My notation/preferred definitions may vary from his, but there are no new ideas here— writing this was mostly an exercise for myself.

The Continuous Heisenberg Group

The **Heisenberg group** H is the group structure on $\mathbb{R}^3$ given by

$$(a, b, c)(x, y, z) = (a + x, b + y, c + z + \frac{1}{2}(ay - bx)).$$

This is a Lie group with Lie algebra $\mathfrak{h}$ spanned by X, Y, Z with the lone nontrivial relation given by $[X, Y] = Z$. The concrete realization of X, Y, Z as left-invariant vector fields is given by

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z}.$$

We will further define Z_n to be $\frac{1}{n}Z$; now let d_n be the Riemannian distance on H for which X, Y, Z_n is orthonormal, and observe that $\frac{1}{n}d_1$ is the Riemannian distance on H for which nX, nY, nZ are orthonormal.

Here is the first hint of some interesting homogeneity in H.

Lemma 1. *The Riemannian manifolds* (H, d_n) *and* $(H, \frac{1}{n}d_1)$ *are isometric.*

Proof. Let $\delta(x, y, z) = (x/n, y/n, z/n^2)$. We claim that δ is an isometry $(H, \frac{1}{n}d_1) \to (H, d_n)$.

To show that δ is an isometry, it suffices to show that $d\delta_p(nX)$ is $X_{\delta(p)}$, and similarly for Y and Z. Indeed,

$$d\delta_p(nX) = nX_p(x/n)\frac{\partial}{\partial x} + nX_p(z/n^2)\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - \frac{p_2}{2n}\frac{\partial}{\partial z} = X_{\delta(p)}$$

$$d\delta_p(nZ) = nZ_p(z/n^2)\frac{\partial}{\partial z} = \frac{1}{n}\frac{\partial}{\partial z} = (Z_n)_{\delta(p)}.$$

And replicating the calculation for X for Y shows that $d\delta_p(nY) = Y_{\delta(p)}$. □

Noting that d_1 is the "standard" Riemannian metric on H, the metric $\frac{1}{n}d_1$ has a special name: it is the **dilate** of d_1 by n. There is a precise way to make formal the idea of taking the limit of $\frac{1}{n}d_1$ as $n \to \infty$ given by *Gromov-Hausdorff limits*. Here is the relevant definition: let X, Y be metric spaces, and consider a subset $\sim \subseteq X \times Y$ so that:

1. The projection of R to each factor is an ϵ–net in the relevant distance.
2. If $x \sim y$ and $x' \sim y'$, then $d_X(x, x')$ and $d_Y(y, y')$ are ϵ–close.

Such a subset is called an **ϵ–relation between X and Y**, and one can check that

$$D_{GH}(X, Y) = \inf\{\epsilon : \exists \text{ an } \epsilon\text{–relation between } X \text{ and } Y\}$$

is a metric on the family of isometry classes of compact metric spaces, known as the Gromov–Hausdorff distance. Given pointed (not necessarily compact) metric spaces (X_n, x_n), (Y, y) we define the **pointed Gromov–Hausdorff convergence** $X_n \xrightarrow{GH} Y$ to mean that for all $R > 0$,

$$\lim_{n\to\infty} D_{GH}(\overline{B}_{X_n}(x_n, R), \overline{B}_Y(y, R)) = 0;$$

i.e., for each $R > 0$, the closed R–balls in the X_n's based at x_n converge Gromov–Hausdorffly to the closed R–ball in Y based at y.

Given a metric space (X, d), a sequence of positive numbers $\epsilon_n \to 0$, and a sequence of basepoints x_n in X, we define the **asymptotic cone** $\mathrm{Cone}(X, x_n, \epsilon_n)$ to be (if it exists) the pointed Gromov–Hausdorff limit of the pointed metric spaces $(X, x_n, \epsilon_n d)$. This allows us to state the titular result of this write-up.

Theorem 1. *Let $\epsilon_n \to 0$ be any sequence of positive numbers. Then* $\mathrm{Cone}(H, 0, \epsilon_n) = (H, d_{cc})$.

To prove Theorem 1, we need to do two things: (i) show that the scaled metrics converge to d_{cc} pointwise, and (ii) show that the relevant closed balls are well-behaved enough to strengthen (i) to Gromov–Hausdorff convergence. For simplicity of notation, we may assume that $\epsilon_n = 1/n$. We will use the following two technical lemmas.

Lemma 2 (Coarse Equivalence)**.** *For all $R > 0$ we may find a $C = C(R)$ so that if $p, q \in B_{cc}(0, R)$, then*

$$d_n(p, q) \leqslant d_{cc}(p, q) \leqslant d_n(p, q) + \frac{C}{\sqrt{n}}$$

Now let exp be the exponential map, so that $\exp(a, b, c) = \exp(aX + bY + cZ)$, and let $\exp_p = L_p \exp$.

Lemma 3 (Ball–Box Theorem)**.** *There is a universal $C > 1$ so that for every $p \in H$,*

$$B_{cc}(p, r/C) \subseteq \exp_p(\mathrm{Box}(r)) \subseteq B_{cc}(p, Cr)$$

Moreover, fixing $R > 0$, there are constants C', r (possibly depending on R) so that $B_{cc}(p, Cr) \supseteq B_{d_1}(p, C'r^2)$ for every $p \in B_{cc}(0, R)$. On compacta there is some K for which $d_{cc} \leqslant K\sqrt{d_1}$.

Before proving them, we will explain how they furnish a proof of Theorem 1.

Proof of Theorem 1. Fix $R > 0$. Observe that by Lemma 3, there is an $R' > 0$ so that

$$B_{cc}(0, R') \supseteq B_1(0, R) \supseteq B_2(0, R) \supseteq \cdots \supseteq B_{cc}(0, R),$$

and further, by Lemma 2, $d_n \to d_{cc}$ uniformly on $B_{cc}(0,R)$. From now on, set $B_n = B_n(0,R)$, and $B_\infty = B_{cc}(0,R)$. Pick N so large so that $\|d_n - d_{cc}\|_\infty < \epsilon/3$ on $B_{cc}(0,R')$, and $C/\sqrt{n} < \epsilon$ for all $n \geqslant N$. We claim that

$$\{(x,y) : y \in B_\infty, d_{cc}(x,y) < \epsilon/3\}$$

is an ϵ-relation on $B_n \times B_\infty$ for all $n \geqslant N$. First we check the comparison property: if $x \sim y$ and $x' \sim y'$, then $d_n(x,x')$ is $\epsilon/3$-close to $d_{cc}(x,x')$. But as

$$d_{cc}(x,x') \leqslant d_{cc}(x,y) + d_{cc}(y,y') + d_{cc}(y',x'),$$

since $x \sim y$ and $x' \sim y'$, we have that $d_{cc}(x,x')$ is $2\epsilon/3$-close to $d_{cc}(y,y')$. Thus $d_n(x,x')$ is indeed ϵ-close to $d_{cc}(y,y')$.

To see that $\sim$ is ϵ-dense, we need only check onto the first factor. The estimate in Lemma 2 tells us that $B_n \supseteq B_\infty \supseteq B_n(0, R - C/\sqrt{n})$, so that for any $z \in B_n$, we may bound the distance $d_n(z, B_\infty) \leqslant C/\sqrt{n} < \epsilon$ since n is sufficiently large. As the projection of $\sim$ on B_n includes every $x \in B_\infty$, it follows that any $z \in B_n$ is in the d_n ϵ-neighborhood of some $x \in B_\infty$; i.e., $\sim$ is ϵ-dense. □

We should now prove Lemma 2.

Proof of Lemma 2. The upper bound is what we need to show. The key here is the presentation given above of H: in particular, we have that the first two coordinates of H act as $\mathbb{R}^2$. Thus given a curve γ with $\gamma(0) = (p,q,r)$ and $\dot\gamma = aX + bY + cZ$, then

$$\gamma^1(t) = p + \int_0^t a(x)\,dx, \qquad \gamma^2(t) = q + \int_0^t b(x)\,dx. \tag{1}$$

This ends up allowing for a "orthogonal projection" of curves in the following sense: let $x, y \in \mathbb{R}^3$, and consider a smooth (in the Heisenberg geometry) curve γ from $x = (p,q,r)$ to y. Notate $\dot\gamma$ as above, and let σ be the unique curve so that $\sigma(0) = x$ and $\dot\sigma = aX + bY$. We claim that the unique curve η such that $\eta(0) = \sigma(1)$ and $\dot\eta = cZ$ has endpoint $\eta(1) = q$.

Indeed, by the calculation in (1), it follows that $\gamma^1(t) = \sigma^1(t)$ and $\gamma^2(t) = \sigma^2(t)$ for all t. Thus we calculate directly using the Heisenberg group law that

$$\sigma(t)^{-1}\gamma(t) = (0, 0, \gamma^3 - \sigma^3)$$

Thus $\frac{d}{dt}(\sigma(t)^{-1}\gamma(t)) = cZ$, and thus $L_{\sigma(1)}L^{-1}_{\sigma(t)}\gamma(t)$ is a curve with derivative cZ, and initial point $\sigma(1)$. This proves that

$$L_{\sigma(1)}L^{-1}_{\sigma(t)}\gamma(t) = \eta(t)$$

for all t, and so $\eta(1) = y$. This gives us a "triangle" of curves by which we may bound the d_{cc}-distance from x to y.

Now suppose without loss of generality that $x, y \in B_{cc}(0,R)$, and γ is a d_n-length-minimizing curve; i.e., $\operatorname{len}_{d_n}\gamma = d_n(x,y)$. By observing that $\dot\sigma$ has no Z part, it follows that $\operatorname{len}_{d_{cc}}\sigma \leqslant \operatorname{len}_{d_n}\gamma$, so in total we have that

$$d_{cc}(x, \sigma(1)) \leqslant \mathrm{len}_{d_{cc}} \sigma \leqslant \mathrm{len}_{d_n} \gamma = d_n(x, y). \tag{2}$$

To bound $d_{cc}(\sigma(1), y)$, we first bound $\mathrm{len}_{d_1} \eta$, and then use Lemma 3 (the Ball-Box Theorem).

Observe that

$$\mathrm{len}_{d_1} \eta = \int_0^1 \|a(t)Z\|_{d_1} \, dt = \int_0^1 |a(t)| \, dt$$

but since we can compare $\|Z\|_{d_1}$ and $\|Z\|_{d_n}$ and can bound $\mathrm{len}_{d_n} \eta$:

$$\int_0^1 \|a(t)Z\|_{d_n} \, dt \leqslant \mathrm{len}_{d_n} \gamma = d_n(x, y) \leqslant d_{cc}(x, y) \leqslant 2R$$

it then follows that

$$\mathrm{len}_{d_1} \eta = \int_0^1 |a(t)| \, dt = \int_0^1 \frac{1}{n} \|a(t)Z\|_{d_n} \, dt \leqslant \frac{2R}{n},$$

and then by Lemma 3, there is a universal constant K (possibly depending on R) so that

$$\underbrace{d_{cc}(\sigma(1), y) \leqslant K\sqrt{d_1(\sigma(1), y)}}_{\text{Lemma 3}} \leqslant K\sqrt{\mathrm{len}_{d_1} \eta} \leqslant \frac{K\sqrt{2R}}{\sqrt{n}}. \tag{3}$$

The triangle inequality then yields for $C = K\sqrt{2R}$,

$$d_{cc}(x, y) \leqslant d_{cc}(x, \sigma(1)) + d_{cc}(\sigma(1), y) \leqslant \underbrace{d_n(x, y)}_{(2)} + \underbrace{C/\sqrt{n}}_{(3)}$$

as desired. □

Now we need to prove Lemma 3.

Proof of Lemma 3. Since d_{cc} is left-invariant, it suffices to show this is true at the identity. Since d_{cc} induces the manifold topology, it follows that there is some $r_0, r_1 > 0$ so that

$$B_{cc}(0, r_0) \subseteq \exp(\mathrm{Box}(r_1)) \subseteq B_{cc}(0, 1)$$

Let δ_t be the dilation on H; then by the definition of the CC distance, δ_t commutes with B_{cc}, and a calculation shows it does for $\exp \mathrm{Box}$ as well. Thus

$$B_{cc}(0, tr_0) \subseteq \exp(\mathrm{Box}(tr_1)) \subseteq B_{cc}(0, t)$$

for all t, proving the first part of the theorem.

Next, observe that $\exp_p$ is locally bilipschitz from the Euclidean metric on the tangent space to the Riemannian metric on H, say of constant C'. Moreover, we have that for all small $r < 1$, $B_E(0, r^2) \subseteq \mathrm{Box}(r)$, so picking r small enough so that $B_E(0, r^s)$ is inside of the bilipshitz neighborhood of 0 in

$T_p(H)$:

$$B_{d_{cc}}(p, Cr) \supseteq \exp_p(\mathrm{Box}(r)) \supseteq \exp_p(B_E(0, r^2)) \supseteq B_{d_1}(p, C'r^2).$$

Thus on a compact set X, there is a uniform C' and r sufficiently small so that

$$B_{cc}(p, Cr) \supseteq B_{d_1}(p, C'r^2) \tag{4}$$

for all p in X. This implies that $1_X : (X, d_1) \to (X, d_{cc})$ is locally (1/2)-Hölder continuous, so all that remains is to check that it is *globally* Hölder of the same exponent on X. Indeed, if not then for all n we may find an x_n, y_n so that

$$d_{cc}(x_n, y_n) > n d_1(x_n, y_n)^{1/2}. \tag{5}$$

By compactness (in both metrics!) we may extract convergent subsequences for both d_1 and d_{cc}. Suppose that $x_{n_k} \to x$ and $y_{n_k} \to y$, and further x, y are far apart by local Hölder continuity. Thus taking the limit of (5) implies that $d_{cc}(x, y) = \infty$, which is impossible.

Hence there is some constant K for which $d_{cc} \leqslant K\sqrt{d_1}$ on X, as desired.

□

The Asymptotic Cone of a Carnot Group

The above calculation relied in an essential way on the coordinate system for the Heisenberg group. To prove a more general theorem, we need a technical theorem.

Lemma 4 (Grönwall Lemma for Lie Groups). *Suppose G is a Lie group, $\| - \|$ a norm on $\mathfrak{g}$, and d a left-invariant Riemannian distance on G. Fix $\nu > 0$. Then there is a $C = C(\nu) > 0$ so that if two absolutely continuous curves σ, γ have speed bounded by ν (i.e., $\|\sigma'\|, \|\sigma'\| < \nu$) a.e. and there is some $\epsilon > 0$ so that $\|\sigma' - \gamma'\| < \epsilon$ a.e., then*

$$d(\sigma(1), \gamma(1)) \leqslant C\epsilon.$$

Now we can prove the general theorem.

Theorem 2. *Suppose G is a Lie group, Δ is a bracket-generating left-invariant distribution. If $\Delta^\perp$ satisfies $\Delta_1 \oplus \Delta_1^\perp = \mathfrak{g}$, and $\langle -, - \rangle_n$ is a sequence of left-invariant Riemannian metrics so that*

1. *$\Delta_1 \perp \Delta_1^\perp$ in $\langle -, - \rangle_n$ for all n,*
2. *$\|X\|_n = \|X\|_0$ for all n,*
3. *$\|X\|_n \to \infty$ as $n \to \infty$ for all $X \notin \Delta$.*

Letting d_{cc} be the sub-Riemannian distance associated with $\langle -, - \rangle_0$ and Δ, and d_n the Riemannian distance associated with $\langle -, - \rangle_n$, then for all $R > 0$ there is an infinitesimal sequence ϵ_n so that

$$d_n(p, q) \leqslant d_{cc}(p, q) \leqslant d_n(p, q) + \epsilon_n$$

on $B_{cc}(1, R)$.

Proof. This theorem follows the scheme from the previous section. The first inequality is trivial, so we must show the second. To do so, we will break up a d_n-length-realizing curve into a curve parallel to Δ and a curve parallel to $\Delta^\perp$, and bound each part.

First, as the d_0-unit-sphere S in $\Delta_1^\perp$ is compact, it follows that for each n, we have some K_n to so that $\inf_S \|Z\|_n \geqslant K_n$. Thus there is a diverging sequence K_n so that for all $Z \in \Delta_1^\perp$, we have $K_n\|Z\|_0 \leqslant \|Z\|_n$.

Now fix some $R > 0$, and let $p, q \in B_{cc}(1, R)$ (in particular, this implies $p, q \in B_{d_n}(1, R)$). Let γ_n be a constant-speed curve from p to q whose length is $d_n(p, q)$; then $\|\dot\gamma\|_n < 2R$. Set γ' to be the pullback of $\dot\gamma$ to $\mathfrak{g}$ writing $\gamma' = X(t) + Z(t)$, we have that $X \perp Z$ in $\langle -, -\rangle_n$, so we may compute

$$K_n\|Z\|_0 \leqslant \|Z\|_n \leqslant \|X + Z\|_n = \|\dot\gamma\|_n < 2R.$$

We thus have that $\|Z\|_0 \leqslant 2R/K_n$ is relatively short for all t.

Now if we set σ to be the unique curve with $\sigma(0) = \gamma(0)$ and $\dot\sigma = L_{\sigma(t),*}X$, we have that $\|\sigma' - \gamma'\|_0 = \|Z\|_0$. Applying Grönwall's lemma, it follows that $d_0(\sigma(1), q) \leqslant 2CR/K_n$ for some C.

Now we can write

$$d_{cc}(p, q) \leqslant d_{cc}(p, \sigma(1)) + d_{cc}(\sigma(1), q).$$

We can bound $d_{cc}(p, \sigma(1))$ by the d_{cc}-length of σ. But this is the d_n-length of σ, which is bounded above by the d_n-length of γ, which we chose to be distance-realizing between p and q. Thus $d_{cc}(p, \sigma(1)) \leqslant d_{cc}(p, q)$.

To bound the second term, we need to compare d_{cc} and d_0, so we may apply our use of Grönwall's lemma. Set $\xi(r) = \operatorname{diam}_{d_{cc}}(\overline{B}_{d_0}(1, r))$. Clearly ξ is increasing and $d_{cc} \leqslant \xi d_0$, and as both d_0 and d_{cc} are distances which induce the (proper) Lie group topology, it follows that $\xi \to 0$ as $r \to 0$.

Then $d_{cc}(\sigma(1), q) \leqslant \xi(d_0(\sigma(1), q))$, and so by the Grönwall bound, we have this this is at most $\epsilon_n = \xi(2CR/K_n)$, where $\epsilon_n \to 0$. $\square$

Applying the same reasoning as in the first section:

Corollary 1. *In the notation of Theorem 2, the Gromov-Hausdorff limit of* (G, d_n) *is* (G, d_{cc}).

Now let G be a Carnot group, with Carnot algebra $\mathfrak{g}$; let $\langle -, -\rangle$ be the left-invariant Riemannian metric relative to a fixed stratification $\{V_i\}$ (so that the strata are orthogonal), and let d_{cc} be the according Carnot metric. Let $X_1, \ldots, X_m$ be an orthonormal basis. Give G exponential coordinates $x_1, \ldots, x_m$, and let $\delta_{1/n}$ take x_i to $n^{-j}x_i$, where X_i is in the jth stratum. This is smooth map $G \to G$, and we may compute that

$$d\delta_{1/n}(nX_i) = \frac{1}{n^{j-1}}X_i.$$

Thus if $\{n^{j-1}X_i\}$ is orthonormal in $\langle -, -\rangle_n$, it follows that the Riemannian metric induced by $\langle -, -\rangle_n$ is isometric to $n^{-1}d$.

Note that $\langle -, -\rangle_n$ satisfies the requirements of Theorem 2, with the "stable distribution" being the first stratum V_1 of $\mathfrak{g}$. We may then apply Corollary 1 to obtain:

Corollary 2. *Suppose G is a Carnot group with left-invariant Riemannian metric d adapted to a stratification of $\mathfrak{g}$. Then if d_{cc} is the Carnot-Caratheodory distance supported on $\mathfrak{g}$'s first stratum, the Gromov-Hausdorff limit of $(G, n^{-1}d)$ is (G, d_{cc}). That is, (G, d_{cc}) is G's asymptotic cone.*