When are one-parameter subgroups geodesics?

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1 Introduction

There are two "exponential mappings" floating around in the world of Lie groups: the Lie algebra exponential, and the Riemannian exponential at the origin of G. Despite having the same name, they often do not agree. The goal of this document is to outline some equivalent conditions for when they agree.

2 Background

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra, and let $\langle -, - \rangle$ be an inner product (in fact, one can assume it is just a non-degenerate symmetric bilinear form) on $\mathfrak{g}$. We may then endow G with the unique Riemannian metric $\langle -, - \rangle$ which is left-invariant: $\langle -, - \rangle_g = \langle (dL_{g^{-1}})_g -, (dL_{g^{-1}})_g - \rangle_e$, and $\langle -, - \rangle_e$ is the given inner product. Denote by ∇ the associated Levi-Civita connection; i.e, if $\mathfrak{X}(G)$ is the set of vector fields on G, then $\nabla : \mathfrak{X}(G) \times \mathfrak{X}(G) \to \mathfrak{X}(G)$ is the unique function satisfying $\nabla_X Y - \nabla_Y X = [X, Y]$ (the Lie bracket of vector fields) and $Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$. Note however that when $\langle -, - \rangle$ is left invariant, and X, Y, Z are left invariant vector fields, then $[X, Y]$ is the Lie bracket of $\mathfrak{g}$, and $Z\langle X, Y \rangle = 0$; moreover, $\nabla_X Y$ is also left-invariant, so ∇ may be taken as a function $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then by combining the two above conditions, we obtain the *Koszul formula*: if $X, Y, Z \in \mathfrak{g}$, then

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \sum_{cyc} \langle [X, Y], Z \rangle.$$

Given a complete Riemannian manifold M and $p \in M$ and $v \in T_p M$, the local solution γ to the ordinary differential equation $\gamma(0) = p$ and $\gamma'(0) = v$ may be extended to a geodesic line $\gamma_v : \mathbb{R} \to M$. If M is also a Lie group, one may ask: when is this a homomorphism?

Proposition 1 (Helgason, Prop II.1.4). *Let $X, Y \in \mathfrak{g}$. Then $t \mapsto \gamma_X(t)$ is a smooth homomorphism if and only if $\nabla_X X = 0$.*

Proof. We extend X to its associated left-invariant vector field $\widetilde{X}$. As left-invariant metrics on Lie groups are complete, we have the curve γ_X so that $\gamma_X(0) = 1_G$ and $\gamma_X'(s) = \widetilde{X}_{\gamma(s)}$. Since $\nabla_X X = 0$, it follows that γ_X is a geodesic curve, and in fact is the unique geodesic with origin 1_G and starting velocity X.

Now we consider the two curves $t \mapsto \gamma_X(s+t)$ and $t \mapsto \gamma_X(s)\gamma_X(t)$. These curves have starting velocities $\gamma_X'(s)$ and $(dL_{\gamma_X(s)})_e X$, respectively, and as $\gamma_X'(s) = \widetilde{X}_{\gamma(s)} = (dL_{\gamma_X(s)})_e X$, it follows that these two curves are both geodesic with equal starting positiions and velocities. Thus they are the same.

Conversely, if θ is a smooth homomorphism with starting velocity X, then one readily calculates that $\theta(0) = e$ and $\theta'(s) = \widetilde{X}_{\theta(s)}$ for all s. In particular, if γ_X is a homomorphism, then this implies that $\nabla_X X = 0$. $\square$

This provides a characterization for when a given one-parameter subgroup is a geodesic (and vice-versa). Later we will see a characterization for when they all are.

Observe that the preceding proof works for any affine connection ∇ of G, not necessarily a Levi-Civita connection for a Riemannian metric; in particular, for each $X \in \mathfrak{g}$ we may choose an affine connection for which $\nabla_X X = 0$, so that the resulting geodesic γ_X is a smooth homomorphism $\mathbb{R} \to G$. We have thus found:

Corollary 1. *For each $X \in \mathfrak{g}$, there is a unique homomorphism* $\exp tX : \mathbb{R} \to G$ *so that* $\exp(0) = 1_G$ *and* $\exp'(0) = X$*. This is called the* **Lie algebra exponential**.

If G is a matrix Lie group, then $\mathfrak{g}$ is also given by matrices, and so the well-defined $\exp tX = \sum_{i=0}^{\infty} (tX)^n/n!$ is indeed a smooth homomorphism $\mathbb{R} \to G$. The map $X \mapsto \exp X$ is smooth, and has derivative I at 0 in $\mathfrak{g}$. Thus the inverse function theorem tells us that there is a neighborhood U of 0 in $\mathfrak{g}$ for which $X \mapsto \exp X$ is a diffeomorphism onto its image; on the other hand, it is generally not true that $X \mapsto \exp X$ is surjective onto G (this is not even the case for $\mathrm{SL}_2(\mathbb{R})$). The utility of this local diffeomorphism is that open neighborhoods of 1_G generate G, so it holds that $\exp(U)$ generates G (even if $\exp(\mathfrak{g}) \neq G$).

Given $g \in G$, there is a smooth homomorphism $G \to G$ given by $c_g(x) = gxg^{-1}$. The derivative of c_g takes $\mathfrak{g} \to \mathfrak{g}$, and is known as the **adjoint mapping** Ad_g. There is a useful formula relating the adjoint mapping and exp; if $X \in \mathfrak{g}$, then

$$\mathrm{Ad}_{\exp X} Y = \sum_{i=0}^{\infty} \frac{[X, \ldots, X, Y]}{n!} := e^{\mathrm{ad}_X} Y.$$

Note that this implies that

$$\frac{d}{dt} \mathrm{Ad}_{\exp tX} = \frac{d}{dt} e^{\mathrm{ad}_{tX}} = \frac{d}{dt} \sum_{i=0}^{\infty} \frac{t^n [X, -]^n}{n!},$$

where $[X, -]^n$ is iterating the Lie bracket n times. As such, we have another useful formula relating the group operation and the Lie bracket:

$$\frac{d}{dt} \mathrm{Ad}_{\exp tX} \bigg|_{t=0} = [X, -].$$

3 A Characterization

Here is the nice fact.

Theorem 1 (Helgason, Exercise A.6, II). *Suppose G is connected with Lie algebra $\mathfrak{g}$, and let $\langle -, - \rangle$ be a left-invariant Riemannian metric (or a nondegenerate symmetric bilinear form), with associated Levi-Civita connection ∇. The following are equivalent.*

1. *The geodesics through e are the one-parameter subgroups.*
2. *$\langle X, [Y, X] \rangle = 0$ for all $X, Y \in \mathfrak{g}$.*
3. *$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$ for all $X, Y, Z \in \mathfrak{g}$.*
4. *$\langle -, - \rangle$ is right-invariant.*
5. *$\langle -, - \rangle$ is invariant under $d(g \mapsto g^{-1})$.*

Such a metric satisfying any of these equivalent conditions is called **bi-invariant**.

Proof. First observe that (1.) is equivalent by Proposition 1 to the condition that $\nabla_X X = 0$ for all $X \in \mathfrak{g}$. The skew-symmetry of the Lie bracket and the Koszul formula imply that

$$\langle X, [Y, X] \rangle = 2 \langle \nabla_X X, Y \rangle;$$

this immediately gives that (1.) is true if and only if (2.) is.

If (2.) is true, then we have in particular that $\langle X + Z, [Y, X + Z] \rangle = 0$ for all $X, Y, Z \in \mathfrak{g}$. Expanding this out we obtain $\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$, so (2.) implies (3.), and by setting $Z = X$, (2.) follows from (3.).

For the next equivalence, first observe that if $\langle -, - \rangle$ is Ad_g-invariant for all $g \in G$, then by left-invariance it must also be right-invariant, and conversely a bi-invariant metric is Ad-invariant. Thus it suffices to show that (3.) is equivalent to Ad-invariance. Moreover, it suffices to Ad_g-invariance for $g \in \exp \mathfrak{g}$, as $\exp \mathfrak{g}$ generates G and $\mathrm{Ad}_{(-)} : G \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism.

One readily calculates that

$$\frac{d}{dt} \langle \mathrm{Ad}_{\exp tX} -, \mathrm{Ad}_{\exp tX} - \rangle \bigg|_{t=0} = \langle [X, -], - \rangle + \langle -, [X, -] \rangle = \langle -, [X, -] \rangle - \langle [-, X], - \rangle, \qquad (*)$$

and as $\langle \mathrm{Ad}_e X, \mathrm{Ad}_e Y \rangle = \langle X, Y \rangle$, it follows that $\langle -, - \rangle$ is Ad-invariant if and only if it is constant, which is true if and only if $(*) = 0$; i.e., (3.) holds.

The last equivalence between (4.) and (5.) is similar to the equivalence with Ad-invariance: follows from the formula

$$d(h \mapsto h^{-1})_g = -(dR_{g^{-1}})_e (dL_{g^{-1}})_g,$$

and the fact that $\langle -, - \rangle$ is already left-invariant.

□