Isometries on Proper Spaces are Proper

William Jones

Here is a nice fact that involves two uses of the same word agreeing.

Proposition 1. *Suppose X is a connected proper metric space. Then the action of* $\operatorname{Isom} X$ *is proper.*

We should say something about properness here.

- A *map* $f : X \to Y$ between topological spaces is proper if $f^{-1}(\text{compact})$ is compact.
- A *metric space* X is proper if for all $x \in X$, the map $d(x, -) : X \to \mathbb{R}$ is proper. That is, a metric space is proper if and only if closed balls are compact.
- The *continuous action* of a topological group G on a space X is proper when the map $(g, x) \mapsto (x, gx)$ is proper. That is, for each pair of compact sets K, C in X, we have that $\{g : gK \cap C \neq \varnothing\}$ is compact in G.

The isometry group of X is given the compact-open topology, with which it acts continuously on X— so it is not nonsense to describe the action as proper or non-proper. The relevant theorem in proving Proposition 1 is the following well-known compactness criterion for the space of continuous maps. Call a set F of maps $X \to Y$ *pointwise precompact* when $\{f(x) : f \in F\}$ is precompact in Y for each $x \in X$.

Theorem 1 (Arzela-Ascoli)**.** *Suppose X is a separable metric space and Y is a metric space. Endow* $\operatorname{Map}(X, Y)$ *with the topology of compact convergence; then pointwise precompact and equicontinuous subsets of* $\operatorname{Map}(X, Y)$ *are metrizable, and are compact when closed.*

Proof. Suppose S is a subset of $\operatorname{Map}(X, Y)$. Pick a countable dense subset D of X, and for each $d \in D$, let $S(d)$ denote the set $\{f(d) : f \in S\}$. If S is pointwise precompact, then for each $d \in D$, we have that $\overline{S(d)}$ is compact in Y. The countable product of metric spaces is a metric space, and the product of compact spaces is compact, so it follows that $\prod_{d \in D} \overline{S(d)}$ is a compact metric space. Define a map $\Phi : S \to \prod_{d \in D} \overline{S(d)}$ by the rule $\Phi(f) = (f(d))_{d \in D}$.

If S is also equicontinuous, then one can show that the subspace topology of compact convergence is the same as the subspace topology of pointwise convergence over D; i.e., Φ is a homeomorphism onto its image. But the subspace topology of a metrizable space is metrizable, so S's subspace topology of pointwise convergence is metrizable. This proves the first part of the theorem.

Now further suppose that S is closed in $\operatorname{Map}(X, Y)$. Then every convergent sequence in S converges to something in S; this means that every convergent sequence in $\Phi(S)$ as a subspace of $\prod_{d \in D} \overline{S(d)}$

converges to something in $\Phi(S)$ (their topologies are the same, and determined by sequences due to metrizability). But as $\prod_{d\in D}\overline{S(d)}$ is a metric space, this means that $\Phi(S)$ is closed in $\prod_{d\in D}\overline{S(d)}$, and as $\prod_{d\in D}\overline{S(d)}$ is compact so is $\Phi(S)$, hence so is S. $\square$

Observe now that $\mathrm{Isom}(X)$ is an equicontiuous closed subspace of $\mathrm{Map}(X, X)$, so to show that subsets S of $\mathrm{Isom}(X)$ are compact, we must show that (i) X is separable, (ii) S is closed, and (iii) S is pointwise precompact. We will do so in a series of lemmas.

Lemma 1. *Proper metric spaces are separable.*

Proof. Any metric space is the ascending union of balls of (positive) integer radii, all centered at some point. Thus if X is proper, it follows that X is the ascending union of countably many compact sets $\{K_i\}$. Since compact sets are separable, it follows that the union of the countable dense subsets of each K_i is a countable dense subset of X. $\square$

Lemma 2. *Let K, C be compact subsets of X. The set $S(K, C) = \{g : gK \cap C \neq \varnothing\}$ is closed in* $\mathrm{Isom}\, X$.

Proof. By Lemma 1 and Theorem 1, we know that the topology on $\mathrm{Isom}\, X$ is metrizable. Thus it suffices to show that $S(K, C)$ is closed under pointwise convergence. Indeed, if $\{g_i\}$ is a convergent sequence in $S(K, C)$, which converges to $g \in \mathrm{Isom}\, X$, we must find some $k \in K$ so that $g(k) \in C$. For each i we may find a $k_i \in K$ so that $g_i(k_i) \in C$. Pick a convergent subsequence k_{i_j} of the k_i converging to $k \in K$. Then for each j, we have that $g_{i_j}(k_{i_j})$ is in C, and as $g_{i_j}(k_{i_j}) \to g(k)$, it follows that $g(k) \in C$. $\square$

We call a family S of maps *precompact at a point* of there is some x_0 for which $\{f(x_0) : f \in S\}$ is precompact. This seemingly weaker condition is in fact equivalent to pointwise precompactness under the relevant hypotheses.

Lemma 3. *Suppose that X is a connected and proper metric space. Then if a family S of maps $X \to X$ is uniformly equicontinuous and precompact at a point, then S is pointwise precompact.*

Proof. Let $x_0 \in X$ be a point at which S is precompact. As S is uniformly equicontinuous, there is some $\delta > 0$ so that if $d(a, b) < \delta$, then $d(f(a), f(b)) < 1$ for every $f \in S$ and $a, b \in X$. By connectedness we may find finitely many points $x_1, \ldots, x_N$ so that $x_1 = x$ and $x_N = y$, and for each i we have that $d(x_i, x_{i+1}) < \delta$. Then for all $f \in S$, it follows that $d(f(x_0), f(y)) < N$. In particular, we have that $\{f(y) : f \in S\}$ is bounded, and thus as X is proper is also precompact. $\square$

Lemma 4. *Let X be a proper metric space, and let $S(K, C)$ be as in Lemma 2. The set $S(K, C)$ is precompact at a point.*

Proof. Pick any $k \in K$. Then as every $f \in S(K, C)$ is an isometry, it follows that $f(K)$ has diameter $\mathrm{diam}\, K$, and as $fK \cap C \neq \varnothing$, this implies that $d(f(k), C) < \mathrm{diam}\, K$. It follows that the set $\{f(k) : f \in S(K, C)\}$ is bounded (say by $\mathrm{diam}\, K + \mathrm{diam}\, C$), and therefore precompact. $\square$

Thus by Lemma 3, it follows that $S(K, C)$ is pointwise precompact. For completeness, we spell out how these lemmas assemble to a proof of Proposition 1.

Proof of Proposition 1. By Lemmas 1, 2, 3, and 4, it follows that a connected proper metric space is separable, and for each pair of compact subsets $K, C \subseteq X$, we have that $S(K, C)$ is closed and pointwise precompact in $\operatorname{Isom} X$. As $\operatorname{Isom} X$ is a closed subspace of $\operatorname{Map}(X, X)$, it follows by Theorem 1 that $S(K, C)$ is compact in $\operatorname{Isom} X$. □

One classical application of Proposition 1 is the following.

Proposition 2. *The isometry group of a finite-dimensional Riemannian manifold acts properly.*

Proposition 1 also gives a natural setting to study quasi-isometries, as the Šwarz-Milnor Lemma requires a proper action. Here is an example.

Corollary 1. *Let M be a finite-dimensional Riemannian manifold. If* $\operatorname{Isom} M$ *has a cocompact discrete subgroup, then M is quasi-isometric to a finitely generated group.*

Proof. Suppose $\operatorname{Isom} M$ has a cocompact discrete subgroup Γ. By Proposition 2 we have that the action of $\operatorname{Isom} M$ is proper, and so the action of Γ on M is properly discontinuous. Since Γ is cocompact, the orbit space $\operatorname{Isom} M/\Gamma$ is compact, and as M is a quotient of $\operatorname{Isom} M$ by a compact stabilizer, it follows that the orbit space M/Γ is also compact. Thus Γ acts on M properly discontinuously and cocompactly, so by Šwarz-Milnor is quasi-isometric to M. □