Quasi-Isometry Invariance of b_n Among Nilpotent Groups

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This is a set of notes on one of the applications of the theory developed in [6], as well as some relevant background materials. Namely, we will present a (reasonably) self-contained proof of the following theorem of Shalom.

Theorem (Betti numbers are QI invariant among nilpotent groups). *If G_1 and G_2 are quasi-isometric f.g. nilpotent groups, then*

$$\dim_{\mathbb{R}} H^n(G_1, \mathbb{R}) = \dim_{\mathbb{R}} H^n(G_2, \mathbb{R})$$

for all $n \geqslant 0$.

There are various tools involved; the broad idea is to relate coarse data about groups to the cohomology of their representations, so that groups with finiteness conditions on cohomology will exhibit coarse geometric rigidity.

1 Continuous Cohomology of Locally Compact Groups

Suppose throughout that G is a Hausdorff topological group. To define the continuous cohomology of G with coefficients in a G-module $\pi : G \to GL(E)$, we define the *continuous n-cochains* $C^n_{cts}(G, E)$ to be the G-homogeneous continuous maps $G^n \to E$; i.e., maps $f : G^n \to E$ which are diagonally equivariant $f(gg_1, \dots, gg_n) = \pi(g)f(g_1, \dots, g_n)$. The differential in the chain complex $C^\bullet_{cts}$ is defined as usual:

$$df(g_0, \dots, g_n) = \sum (-1)^i f(g_0, \dots, \widehat{g_i}, \dots, g_n),$$

and the cocycles Z^n_{cts} and coboundaries B^n_{cts} are also defined as usual. We define the *continuous cohomology* of G with coefficients in π to be $H^n_{cts}(G, \pi) = H^n_{cts}(G, E)$ to be cocycles mod coboundaries. In general we do not have that B^n is closed in Z^n, so we may also define the *reduced continuous cohomology* $\overline{H}^\bullet_{cts}(G, \pi)$ to be $Z^n/\overline{B^n}$.

Importantly, if G is discrete then $H^\bullet_{cts}$ is just the "normal" group cohomology. The following fact will be important for proving statements up to quasi-isometry.

Lemma. *Let G be a discrete group and $N \lhd G$ a* ***finite*** *normal subgroup. Then if V is a G-module and a $\mathbb{Q}$-space, then the inclusion $V^N \to V$ of N-fixed vectors induces isomorphisms $H^n(G, V) \cong H^n(G/N, V^N)$ for every $n \geqslant 1$.*

One of the most effective tools in computing cohomology is the *Hochschild-Serre spectral sequence*, which is generally formulated for *discrete* group cohomology. Under some hypotheses, however, the spectral sequence still converges for a non-discrete locally compact group.

Theorem ([2], Théorème 9.1, Hochschild-Serre spectral sequence)**.** *If the following requirements hold,*

1. *G is a locally compact topological group with compact exhaustion,*
2. *Γ is a closed normal subgroup of G,*
3. *there is a Borel section $s : G/\Gamma \to G$ which takes compact sets to precompact sets,*
4. *E is a Fréchet G-module so that $H^\bullet_{cts}(\Gamma, E)$ is a Hausdorff topological space (i.e., $\overline{H^\bullet} = H^\bullet$),*

then $H^\bullet_{cts}(\Gamma, E)$ has the structure of a Fréchet G/Γ-module, and there is a spectral sequence supported in the first quadrant with $E_2^{p,q} = H^p_{cts}(G/\Gamma, H^q_{cts}(\Gamma, E))$ that converges to $H^\bullet_{cts}(G, E)$.

We will not need the full generality of this statement. Instead, we may specialize to the case that G, Γ are Lie groups, and E is a Hilbert space on which G acts unitarily (in particular, this implies that E is Fréchet as a G-module).

Corollary 1. *Suppose G is a Lie group, N a normal Lie subgroup of G, and E is a unitary representation of G. Then if $H^\bullet_{cts}(N, E)$ is Hausdorff (i.e., $\overline{H^\bullet} = H^\bullet$),, we have that $H^\bullet_{cts}(N, E)$ is a Fréchet G/N-module, and $H^p_{cts}(G/N, H^q_{cts}(N, E)) \Rightarrow H^\bullet_{cts}(G, E)$.*

The following is from [3], Ch. I, §7, pg. 39. Consider the standard homogeneous cochain complex, and let g act on cochains by right multiplication $g.f(g_0, \ldots, g_n) = f(g_0 g, \ldots, g_n g)$. This is in fact an automorphism of the whole complex, and in the inhomogeneous cochain complex we have that this is written as $g.f(g_1, \ldots, g_n) = gf(g^{-1}g_1 g, \ldots, g^{-1}g_n g)$. Now define on the homogeneous cochains $S_i^n : C^{n+1}(G, E) \to C^n(G, E)$ by the rule

$$S_i^n f(g_0, \ldots, g_{n-1}) = f(g_0, g_1, \ldots, g_i, g_i g, g_{i+1}, \ldots, g_{n-1} g);$$

then $S^n = \sum_{i=0}^{n-1} (-1)^i S_i^n$ forms a chain homotopy of $C^\bullet$ which takes the G-action to the identity automorphisms. We have thus found:

Lemma 1. *The natural G-action*

$$g.f(g_1, \ldots, g_n) = gf(g^{-1}g_1 g, \ldots, g^{-1}g_n g)$$

on $H^\bullet_{cts}(G, E)$ is trivial.

We will also need some harmonic analysis. In the representation theory of locally compact groups, a representation is said to be *irreducible* if the only invariant subspaces are the trivial ones, and if there is a *closed* invariant subspace W, then $\pi|_W$ is said to be a *subrepresentation*. The crucial background here is one of the many results called Schur's lemma.

Theorem 1 ([1], Theorem A.2.2 (Schur's Lemma))**.** *A unitary representation π of G is irreducible if and only if its centralizer in $L(H_\pi, H_\pi)$ (also called its* commutant*) consists of scalar multiples of 1_H.*

One application which is relevant to inductive arguments is:

Corollary 2 ([1], Corollary A.2.3)**.** *Representations of abelian groups act by scalar multiplication. Irreducible representations of abelian groups are one-dimensional.*

Proof. Suppose π is a representation of G abelian. Then $\pi(G)$ is a subgroup of π's commutant, thus π represents G as scalar multiples of 1_H. If π is irreducible, then $H_\pi \neq 0$. Pick $f \neq 0$ in H_π; then $\langle f \rangle$ is invariant under π, so $H_\pi = \langle f \rangle$. □

Note that in the case that G is not finite, there may be non-irreducible representations which have no irreducible subrepresentations.

For example, ifx λ is the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$, then an irreducible subrepresentation of λ would be one-dimensional by Corollary 2, thus be the span of a nonzero function f. We will show that f cannot exist. Indeed, such a function must be constant: we have

$$\|f\|^2 = \int_{\mathbb{R}} |f|^2 \, dx = \int_{\mathbb{R}} |f(x+y)|^2 \, dx = |\pi(y)|^2 \|f\|^2,$$

so $|\pi| \equiv 1$. But then $f(y) = \pi(y) f(0)$, so $|f| \equiv |f(0)|$, which then implies that f is constant nonzero: a contradiction.

In the finite-dimensional case, we may decompose any reducible representation into a direct sum of irreducible ones. This example shows that this is not possible, but we can get the next best thing: the *Fourier transform*

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x t} F(t) \, dt$$

decomposes $L^2(\mathbb{R})$ as not a direct sum, but a *direct integral* of irreducible representations of $\mathbb{R}$. This holds in general (e.g., see Kirillov) for a locally compact group: any representation π of a locally compact group G may be decomposed into a direct integral of irreducible representations; meaning there is a certain standard measure space (V, μ), a collection of representations π_v, and a G-isomorphism

$$E_\pi \cong \int^{\oplus} \pi_v \, d\mu(v),$$

from E_π to the space of square-integrable functions $V \to \sqcup_{ess} E_{\pi_v}$, such that π_v is irreducible μ-a.e. Note that a direct integral decomposition is *not* in general unique, even up to the appropriate equivalences.

The relations between π and the irreducible representations occurring in a direct integral decomposition of π are subtle. We at least have the following.

Lemma 2. *If π is a representation of lcsc G and $\int^{\oplus} \pi_v \, d\mu(v)$ is a direct integral decomposition of π, then if $\overline{H^\bullet}(G, \pi_v) = 0$ μ-a.e., then $\overline{H^\bullet}(G, \pi) = 0$.*

We will now calculate the cohomology of nontrivial irreducible representations of 1-connected nilpotent Lie groups.

Theorem 2 ([2], Théorème 10.1). *Suppose that G is a 1-connected nilpotent Lie group, and $\pi : G \to U(E)$ is a nontrivial irreducible unitary representation on the Hilbert space E. Then $H^\bullet_{cts}(G, E) = 0$.*

Proof. The proof will be by induction on $\dim G$.

We will treat the abelian case separately, as it will be useful (and also prove the base case). If G is abelian, then π is a nontrivial character. Note that $H^\bullet_{cts}(G, E)$ carries a natural G-action induced from E's, which is just multiplication by $\pi(g)$. From Lemma 1, this action is trivial, so it follows that for all $g \in G$, $\phi \in Z^n$, that $(\pi(g)-1)\phi$ is a coboundary. But as π is nontrivial, there is some g' so that $\pi(g') \neq 1$, so there is a nonzero λ so that $\lambda\phi$ is a coboundary; thus so is ϕ.

Now we consider the general case. If G is nilpotent of step s, then $N = G^{(s)}$ is central in G and nontrivial. Let $\tau = \pi|_N$; our first subcase is that τ is nontrivial. Since N is abelian, τ is just scalar multiplication by Corollary 2, and so we may run the preceding argument to conclude that $H^\bullet_{cts}(N, \tau) = 0$. We may then apply Corollary 1 to this situation; as $H^p_{cts}(G/N, H^q_{cts}(N, E)) = H^p_{cts}(G/N, 0) = 0$, the Hochschild-Serre spectral sequence is degenerate, so $H^\bullet_{cts}(G, E) = 0$ as desired.

Now suppose that τ is trivial. The continuous group cohomology of Lie group with trivial action is the same as the Chevalley-Eilenberg cohomology of its Lie algebra (with trivial action). Thus $H^\bullet_{cts}(N, \tau) = H^\bullet(\mathfrak{n}, \tau)$. But $\mathfrak{n}$ is abelian, so reading off the standard Chevalley-Eilenberg chain complex, we obtain that $H^\bullet(N, \tau) = A^\bullet(\mathfrak{n}, E)$ is the algebra of alternating multilinear n-forms $\mathfrak{n}^n \to E$.

The G/N-module structure on $H^q_{cts}(N, \tau)$ is given by the action

$$[g]f(h_1, \ldots, h_n) = gf(g^{-1}h_1 g, \ldots, g^{-1}h_n g).$$

Since N is central, however, it then follows that the action of G/N on $A^q(\mathfrak{n}, E)$ is just the normal G/N-action on $A^q(\mathfrak{n}, E)$ given by the action on E. Thus as a G/N-module, we have that $H^q(N, \tau) \cong \bigoplus_{\dim A^q} E'$. where E' is the G/N-module E. Thus

$$\bigoplus_{\dim A^q} H^p_{cts}(G/N, E') \cong H^p_{cts}(G/N, \oplus E') \cong H^p_{cts}(G/N, H^q_{cts}(N, E)) \Rightarrow H^\bullet_{cts}(G, E)$$

by Corollary 1, but by induction we have that $H^p_{cts}(G/N, E') = 0$, so the spectral sequence degenerates at the first page and so $H^\bullet_{cts}(G, E) = 0$. □

Corollary 3. *Suppose that G is a 1-connected nilpotent Lie group, and π is a nontrivial (not necessarily irreducible) unitary representation of G. Then $\overline{H}^\bullet_{cts}(G, \pi) = 0$.*

Proof. We may disintegrate π into a family of a.e. irreducible representations π_x. Theorem 1 implies that $\overline{H}^\bullet_{cts}(G, \pi_x) = 0$ a.e., so Lemma 2 implies the same for π. □

In *Kazhdan's Property (T)*, it is proved that lcsc G has Property (T) if and only if $H^1(G, \pi) = 0$ for every irreducible unitary representation π. We may thus conclude another corollary.

Corollary 4. *Let G be 1-connected nilpotent. The only irreducible unitary representation of G with nonvanishing cohomology is the trivial one.*

2 Couplings

In order to leverage the tools of harmonic analysis to relate two groups, we need functions on spaces relevant to the groups. Given groups $\Lambda \leqslant \Gamma$, some clear choices would to consider the function spaces $L^2(G)$, $L^2(H)$, or $L^2(G/H)$. However, it is not immediately clear what functions to consider when Λ and Γ are *a priori* unrelated, or even quasi-isometric.

The first part of the following theorem is attributed to Gromov.

Theorem 3 ([6], Theorem 2.1.2)**.** *Two finitely generated groups Γ, Λ are quasi-isometric if and only if there is a locally compact space X on which both Λ and Γ act continuously, properly, and cocompactly.*

Moreover, after replacing Γ with $\Gamma' = \Gamma \times M$ for some finite group M, we may find such a space X so that the three following properties also hold:

1. *Both actions on X are free,*
2. *The compact fundamental domains of Γ' and Λ are also open,*
3. *$X_{\Gamma'} \subseteq X_\Lambda$.*

The idea here is that if ϕ is a quasi-isometry $\Lambda \to \Gamma$, then the quasi-isometry condition implies that there is only a finite failure of injectivity. Thus setting $\Gamma' = \Gamma \times M$ for M sufficiently large, we may assume ϕ is injective. There are linear estimates F_1, F_2 so that

$$F_1 d_\Lambda \leqslant d_\Gamma \phi \leqslant F_2 d_\Lambda, \tag{1}$$

and we take X to be the space of injective functions $\Lambda \to \Gamma$ satisfying the estimates in (1). This is a topological space in the subspace topology of Γ^Λ, and the action of Γ' and Λ' on X is given by post and pre-composition, respectively. A space X satisfying the conditions of Theorem 3 is called a **topological coupling** of Λ and Γ.

The following is attributed to Gromov. Two discrete countable groups Γ, Λ are called **measure equivalent** (ME) if there is a σ-finite measure space (X, μ) such that the actions $\Gamma \curvearrowright X \curvearrowleft \Lambda$ are eseentially free, μ-preserving, commuting, and with finite-measure fundamental domains. The groups are **uniformly measure equivalent** (UME) for every $\gamma \in \Gamma$, $\lambda \in \Lambda$, there are finite subsets $S_\gamma \subseteq \Lambda, S_\lambda \subseteq \Gamma$ so that

$$\gamma X_\Lambda \subseteq X_\Lambda S_\gamma \text{ and } X_\Gamma \lambda \subseteq S_\lambda X_\Gamma.$$

Such spaces X are called respectively **ME couplings** or **UME couplings**. If the action of $\Gamma \times \Lambda$ on an ME coupling is ergodic, we say that the coupling is ergodic.

Topological couplings will allow for a dictionary between the two actions, and ME couplings will allow for induction of representations.

Indeed, if we have a topological coupling X and $x \in X_\Lambda$, then it is not necessarily the case that $\gamma^{-1}x \in X_\Lambda$ for $\gamma \in \Gamma$. However, the fundamental domain condition and freeness imply that there is a unique $\lambda \in \Lambda$ so that $\gamma^{-1}x\lambda \in X_\Lambda$. Define

$$\alpha : \Gamma \times X_\Lambda \to \Lambda$$

so that $\alpha(\gamma, x)$ is the unique λ such that $\gamma^{-1}x\lambda \in X_\Lambda$. Dually we define $\beta(x, \lambda)$ to be the unique γ so that $\gamma^{-1}x\lambda \in X_\Gamma$. Note that $X_\Gamma \subseteq X_\Lambda$ is equivalent to the equality

$$\alpha(\beta(x, \lambda), x) = \lambda, \forall x \in X_\Gamma, \lambda \in \Lambda.$$

Note that if we identify $X_\Lambda = X/\Lambda$, we do not have that Γ's action on X/Λ is well-defined. However, the cocycles α, β give a way to define an action of Γ (resp. Λ) on X/Λ (resp. $\Gamma \backslash X$). Indeed, the induced action on the quotient spaces are exactly given by

$$\gamma.x = \gamma x \alpha(\gamma^{-1}x) \text{ and } x.\lambda = \beta(x, \lambda)^{-1}x\lambda.$$

Thus these cocycles induced by the coupling mediate the two actions.

It will also be important to study the case that Λ is a cocompact subgroup of not-necessarily discrete Γ. In this case, setting $X = \Gamma$, we have that Γ is a UME coupling of Λ and Γ, and X_Γ is a point g_0. The cocycle α remains the same, but the cocycle β is greatly simplified: if $\lambda \in \Lambda$, the unique $\gamma \in \Gamma$ so that $\gamma^{-1}g_0\lambda = g_0$ is nothing but $g_0\lambda g_0^{-1}$. So we have the equation

$$\beta(g_0, \lambda) = g_0\lambda g_0^{-1} \quad \text{if} \quad X_\Gamma = g_0,$$

and further the action $g_0.\lambda$ of Λ on X_Γ is trivial. As any point in Γ may be taken to be X_Γ, we may define β on all of Γ as $\beta(g, \lambda) = g\lambda g^{-1}$, and similarly the action $g.\lambda = g$.

3 Amenability and Existence of UME Couplings

A locally compact topological group G is **amenable** if there is a functional $m \in L^\infty(G)^*$ satisfying

1. (Unit Norm) $\|m\|_{op} = 1$
2. (Positivity) $f \geqslant 0$ a.e. $\Rightarrow m(f) \geqslant 0$
3. (Left-Invariance) $m(g.f) = m(f)$ for every $g \in G$ and $f \in L^\infty(G)$, where $g.f$ is given by $g.f(x) = f(g^{-1}x)$.

Such a function is called a **mean** on G.

Now suppose that G amenable (with mean m) acts on a compact space X. Fix $x \in X$; then for any $f \in C(X)$, we have that $g \mapsto f(gx)$ is a bounded function, so $m(g \mapsto f(gx))$ is a real number. Define $\phi(f)$ to be this number.

Since $\phi(1_X) = 1$, $\phi(g.f) = \phi(f)$, and $\phi(f) \geqslant 0$ for all $f \in C(X)$, the Riesz representation theorem tells us that f defines a G-invariant Borel probability measure on X. This is the **finite invariant measure** condition for amenable groups.

The next step requires a technical lemma.

Lemma 3. *Suppose $f_i : X \to Y_i$ is a family of functions into topological spaces Y_i, and give X the coarsest topology so that each f_i is continuous.*

Then a net $x_\alpha \to x$ converges in X if and only if $f_i(x_\alpha) \to f_i(x)$ converges for each i.

Now suppose $\mathcal{P}$ is the set of G-invariant probability measures on X. We claim that $\mathcal{P}$ is weak*-closed; indeed, if μ_α is a net of elements in $\mathcal{P}$ which converge to some $\mu \in C(X)^*$, then we have that for every $f \in C(X)$, $\mu_\alpha(f) \to \mu(f)$ by Lemma 3. This is just a convergence of nets in the ordinary topology on $\mathbb{R}$, so it clearly preserves G-invariance, positivity, and as $1_X \in C(X)$, preserves having unit mass. Thus $\mu \in \mathcal{P}$, so all nets in $\mathcal{P}$ converge, so $\mathcal{P}$ is weak*-closed. Since $\mathcal{P}$ is contained in the unit ball of $C(X)^*$, which is weak*-compact, it then follows that $\mathcal{P}$ is weak*-compact and convex. The Krein-Milman theorem then yields an extreme point ν of $\mathcal{P}$, which is precisely a G-invariant *ergodic* probability measure on X.

This is sufficient setup to prove the following existence result.

Theorem 4 ([6],Theorem 2.1.7)**.** *If amenable groups Λ and Γ are quasi-isometric, then for some finite M, there is an ergodic UME coupling X of Λ and $\Gamma \times M$, satisfying $X_{\Gamma\times M} \subseteq X_\Lambda$.*

Proof. By Theorem 3, we have a topological coupling X between Λ and $\Gamma \times M$ (which already satisfies the fundamental domain condition). Then X/Λ is compact, so by the preceding discussion there is an ergodic Γ-invariant probability measure μ on X/Λ. By tessellating X by $X_\Lambda = X/\Lambda$, we may extend μ to a measure $\tilde{\mu}$ all of X in a Λ-invariant fashion, giving X the structure of an ergodic UME coupling satisfying the fundamental domain condition. □

4 Induced Representations and Cohomology

Now let Γ, Λ be discrete groups and (X, μ) an ME coupling of them. Since Γ's fundamental domain is inside of Λ's, Shalom suggests one thinks of Λ as the "small group", and Γ as the "big group." Thus we wish to induce representations from Λ up to Γ.

If $\pi : \Lambda \to U(V_\pi)$ is a unitary representation of Λ, we define

$$L^2(X, V_\pi)^\Lambda = \left\{ f : X \to V_\pi : f(x\lambda) = \pi(\lambda^{-1})f(x) \text{ and } \int_{X_\Lambda} \|f(x)\|^2 \, d\mu(x) < \infty \right\}$$

to be the Λ-equivariant extension of $L^2(X/\Lambda, V_\pi)$. Of course, we can also consider $L^2(X_\Lambda, V_\pi)$ itself, so we define **the Γ-representation** $\mathrm{Ind}_\Lambda^\Gamma \pi$ **induced from Λ to** Γ to be either of the following (equivalent) unitary representations:

- (Coset Model) $L^2(X_\Lambda, V_\pi)$ with the action $\gamma f = \pi(\alpha(\gamma, -))f(\gamma^{-1}.-)$

- (Equivariant Model) $L^2(X, V_\pi)^\Lambda$ with the action $\gamma f = f(\gamma^{-1}.-)$.

It will also be important to consider the case that an arbitrary locally compact second countable $G = \Gamma$ is the big group, and Λ is a cocompact subgroup in G. In this case, we shall set the space X in the ME coupling to be $X = G$.

Consider now either (X, μ) a UME coupling of discrete Γ and Λ, or $X = G = \Gamma$ a UME coupling of Λ cocompact inside of G. We will define a map $I : H^n(\Lambda, \pi) \to H^n_{cts}(\Gamma, \mathrm{Ind}_\Lambda^\Gamma \pi)$.

Given $\omega \in C^{n+1}(\Lambda, V_\pi)$, we define

$$I\omega(\gamma_0, \dots, \gamma_n)(x) = \omega(\alpha(\gamma_0, x), \dots, \alpha(\gamma_n, x)).$$

We claim that $I\omega \in C^{n+1}(\Gamma, \mathrm{Ind}_\Lambda^\Gamma \pi)$. This involves verifying that $I\omega(g_0, \dots, g_n)$ as a function of x is in L^2, that $I\omega$ is diagonally Γ-equivariant, and that as a function of Γ^{n+1}, $I\omega$ is continuous (this last condition is trivial if Γ is discrete).

To verify that $I\omega(g_0, \dots, g_n) \in L^2(X_\Lambda, V_\pi)$, note that as X is a UME coupling for any γ there is a finite set S_γ so that $\gamma X_\Lambda S_\gamma^{-1} \subseteq X_\Lambda$. This implies that for any i, $\alpha(g_i, X_\Lambda)$ is a finite set. Thus there are finitely many sets $S_i^j \subseteq X_\Lambda$ and $\lambda_i^j \in \Lambda$ so that

$$\alpha(g_i, x) = \sum_{j=1}^{n_i} \lambda_i^j \mathbf{1}_{S_i^j}.$$

Partitioning $X_\Lambda = \bigsqcup_{k=1}^N P_k$ along all of the S_i^j's and letting λ_k^i be the value of $\alpha(g_i, -)$ on P_k, it then follows that

$$\|I\omega(g_0, \dots, g_n)(x)\|_2^2 = \int_{X_\Lambda} \omega(\alpha(g_0, x), \dots, \alpha(g_n, x))^2 \, d\mu(x) = \sum_{k=1}^N \mu(P_k)\omega(\lambda_k^0, \dots, \lambda_k^n)^2 < \infty.$$

Diagonal Γ-equivariance follows from Λ-equivariance and the cocycle identities for α; indeed we have that

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, x)\alpha(\gamma_2, \gamma_1^{-1}.x)$$

for all $x \in X_\Lambda$, so we may calculate using this and Λ-equivariance of ω to find that:

$$I\omega(\gamma g_0, \dots, \gamma g_n)(x) = \pi(\alpha(\gamma, x))I\omega(g_0, \dots, g_n)(\gamma^{-1}.x) = \gamma.I\omega(g_0, \dots, g_n)(x).$$

Last, in the case that $\Gamma = G$ and Λ is cocompact in G, we must show that $I\omega$ is a *continuous* function from G^{n+1} to $L^2(X_\Lambda, V_\pi)$. However, as $g'g^{-1} \to 1_G$, it follows that $\mu(\{\alpha(g, x) \neq \alpha(g', x)\}) \to 0$. We have seen the possible values ω can take is finite; this is also true for bounded g, so

$$M = \max_x \{(\omega(\alpha(g, x), \dots, \alpha(g_n, x) - \omega(\alpha(g', x), \dots, \alpha(g_n, x)))^2 : \alpha(g, x) \neq \alpha(g', x)\}$$

is finite. We may conclude that

$$\begin{aligned}\|I\omega(g,g_1,\ldots,g_n)-I\omega(g',g_1,\ldots,g_n)\|_2^2 &= \int_{\{\alpha(g,x)\neq\alpha(g',x)\}}(I\omega(g,g_1,\ldots,g_n)-I\omega(g',g_1,\ldots,g_n))^2\,d\mu(x)\\ &\leqslant \mu(\{\alpha(g,x)\neq\alpha(g',x)\})M,\end{aligned}$$

which goes to zero as $g'\to g$. Thus $I\omega$ is continuous, so I is indeed a map $H^n(\Lambda,\pi)\to H^n_{cts}(G,\operatorname{Ind}_\Lambda^G\pi)$ when G is not necessarily discrete.

One further verifies that $Id=dI$, so we have proved the following.

Lemma. *When discrete Λ,Γ are part of a UME coupling, or when discrete Λ is cocompact in not-necessarily-discrete Γ, the map I induces a map in cohomology.*

$$I : H^n(\Lambda,\pi)\to H^n_{cts}(\Gamma,\operatorname{Ind}_\Lambda^\Gamma\pi)$$

We claim that I is injective.

Theorem ([6], Theorem 3.2.1)**.** *If either the UME coupling satisfies $X_\Gamma\subseteq X_\Lambda$, or Λ is cocompact in Γ, then I is injective on both reduced and non-reduced cohomology.*

Proof. The proof will consist of finding a continuous linear left inverse to I which commutes with the differential; the continuity condition is to guarantee injectivity on reduced cohomology.

Recall that $\beta : X_\Gamma\times\Lambda\to\Gamma$ is the dual cocycle to α. For $\sigma\in C^{n+1}(\Gamma,\operatorname{Ind}_\Lambda^\Gamma\pi)$, we define for discrete Λ,Γ the map

$$T\sigma(\lambda_0,\ldots,\lambda_n)=\int_{X_\Gamma}\sigma(\beta(x,\lambda_0),\ldots,\beta(x,\lambda_n))(x)\,d\mu(x) \tag{I}$$

and when Λ is discrete and cocompact in Γ, the cocycle β simplifies to conjugation. However, we cannot integrate over a point, so instead we integrate over X_Λ.

$$T_{cts}\sigma(\lambda_0,\ldots,\lambda_n)=\int_{X_\Lambda}\sigma(g\lambda_0g^{-1},\ldots,g\lambda_ng^{-1})(g)\,d\mu(g). \tag{II}$$

Note that for (I) the hypothesis that $X_\Gamma\subseteq X_\Lambda$ is used in making sure the integral is well-defined.

We should clarify the notation in (I); what is meant by the second (x) in the integrand is not that we are just integrating $\sigma(\beta(x,\lambda_0),\ldots,\beta(x,\lambda_n))$ over x; the cocycle takes values in $L^2(X_\Lambda,V_\pi)$, so this would only give us an L^2-integrable function. To amend this, the notation

$$\sigma(\beta(x_0,\lambda_0),\ldots,\beta(x_0,\lambda_n))(x_0)$$

means "evaluate the function $\beta(x_0,\lambda_0),\ldots,\beta(x_0,\lambda_n)$ at x_0." The resulting function on X_Γ is integrable due to the fact that $\sigma(\beta(-,\lambda_0),\ldots,\beta(-,\lambda_n))$ takes only finitely many values on X_Λ.

We can calculate Λ-equivariance of $T\sigma$, linearity, commutation with the differential, and the left inverse property $TI=1$ using properties of β, which will then prove the desired results for both functions (I) and (II).

To show Λ-equivariance, we may compute as above that

$$T\sigma(\lambda\lambda_0, \ldots, \lambda\lambda_n) = \int_{X_\Gamma} \pi(\alpha(\beta(x,\lambda),x)\left(\sigma(\beta(x.\lambda,\lambda_0),\ldots,\beta(x.\lambda,\lambda_n))(\beta(x,\lambda)^{-1}.x)\right)\, d\mu(x)$$

and as $X_\Gamma \subseteq X_\Lambda$, we have that $\pi(\alpha(\beta(x,\lambda),x)) = \pi(\lambda)$, so after changing variables we have

$$T\sigma(\lambda\lambda_0, \ldots, \lambda\lambda_n) = \pi(\lambda)\int_{X_\Gamma} \sigma(\beta(x,\lambda_0),\ldots,\beta(x,\lambda_n))(\beta(x.\lambda^{-1},\lambda)^{-1}.(x.\lambda^{-1}))\, d\mu(x)$$

One computes that $\beta(x.\lambda^{-1},\lambda)^{-1}.(x.\lambda^{-1}) = x$, which finishes the desired calculation.

The calculation that $TI = 1$ uses $X_\Gamma \subseteq X_\Lambda$ in a similar way, and linearity and commutativity with d are easy to verify. All that remains is to check continuity, which requires technically different methods for (I) and (II), but both follow from the same principal: continuous cocycles being close imply their integrals are close. $\square$

In the case that Λ is discrete cocompact in Γ, it is known ([3]) that I is surjective on ordinary and reduced cohomology, and in fact an isomorphism on non-reduced cohomology as well. Thus:

Corollary 5 ([6], Theorem 3.2.2)**.** *If Λ is discrete cocompact inside the locally compact, second countable group G, we have*

$$\bar{H}^n(\Lambda,\pi) \cong \bar{H}^n(G, \operatorname{Ind}_\Lambda^G \pi),$$

as well as a continuous isomorphism between nonreduced cohomologies.

5 Property H_T^n and a Proof of Theorem 1

Let G be a (second countable, locally compact) group. We say that G has **property** (H_T^n) if whenever π is a unitary representation of G so that $\bar{H}^n(G,\pi)$ is nonzero, then $1_G \subseteq \pi$.

This property is equivalent to the vanishing of the nth reduced cohomology of non-trivial irreducible representations.

Lemma 4. *The group G has property (H_T^n) if and and only if $\bar{H}^n_{cts}(G,\pi) = 0$ whenever π is non-trivial and irreducible.*

Proof. Suppose G has (H_T^n) and let π be irreducible non-trivial. Then $1_G \neq \pi$, so $\bar{H}^n_{cts}(G,\pi) = 0$.

Conversely, let π be a representation so that $\bar{H}^n(G,\pi) \neq 0$. Applying Lemma 2, it follows that G has an irreducible representation π_x so that $\bar{H}^n(G,\pi) \neq 0$; by the assumption this must be the trivial representation, so $1_G \subseteq \pi$. $\square$

The preceding proof also makes clear why the *reduced* cohomology is being considered. Note that we have already seen an example of groups satisfying this.

Corollary 6. *1-connected nilpotent Lie groups have property (H_T^n) for every $n \geqslant 1$.*

The following lemma is analogous to (T).

Corollary 7. *Suppose Λ is discrete and cocompact in G. If G has property (H_T^n), then so does Λ.*

Proof. Let π be a representation of Λ. We first prove that $1_G \subseteq \operatorname{Ind}_\Lambda^G \pi$ implies that $1_\Lambda \subseteq \pi$. For this proof it is convenient to consider the equivariant model of $\operatorname{Ind}_\Lambda^G \pi$; i.e., the space of L^2-integrable functions on G/Λ satisfying $f(x\lambda) = \pi(\lambda^{-1})f(x)$. Now suppose f is an invariant vector; then for all g, we have that $gf(x) = f(x)$, so that $f(g^{-1}x) = f(x)$. Since $G \curvearrowright G/\Lambda$ is transitive, it follows that $f \equiv v$ is a constant function. But then $\pi(\lambda)v = \pi(\lambda)f(x) = f(x\lambda^{-1}) = v$, so $v \in V_\pi$ is fixed by π. $\square$

Corollary 8. *Every finitely generated nilpotent group has property (H_T^n) for every $n \geqslant 1$.*

Proof. The set of torsion elements in a f.g. nilpotent group is a finite normal subgroup, so by Lemma ?? it suffices to prove that finitely generated torsion-free nilpotent groups have (H_T^n). This follows from Corollary 7 and the Mal'cev completion. $\square$

The following theorem provides the link between the Betti numbers and property (H_T^n).

Theorem 5 ([6], Theorem 4.1.1)**.** *Suppose Γ is a finitely generated amenable group with property (H_T^n). If Λ is finitely generated and quasi-isometric to Γ, so that $b_n(\Lambda)$ is finite, then $b_n(\Lambda) \leqslant b_n(\Gamma)$.*

Proof. Since $b_n(\Lambda)$ is finite, it follows that $H^n(\Lambda, 1_\Lambda)$ is finite-dimensional, so in fact $b_n(\Lambda)$ is just $\dim_{\mathbb{C}} \bar{H}^n(\Lambda, 1_\Lambda)$. Replacing Γ with $\Gamma \times M$, we have that $\Gamma \times M$ still has property (H_T^n) by Lemma ??. It then follows from Theorem 4 that there is an ergodic UME coupling X of $\Gamma \times M$ and Λ so that $X_{\Gamma\times M} \subseteq X_\Lambda$. This is the setting for the injection of Theorem ??, so that

$$\bar{H}^n(\Lambda, 1_\Lambda) \hookrightarrow \bar{H}^n(\Gamma \times M, \operatorname{Ind}_\Lambda^{\Gamma\times M} 1_\Lambda)$$

The $(\Gamma \times M)$-representation induced by 1_Λ is the regular representation on $L^2(X/\Lambda)$. Since integration is continuous on X/Λ, it follows that $L^2(X/\Lambda)$ has the topological direct sum decompostion $\mathbb{C} \oplus L_0^2(X/\Lambda)$, where the latter space is the zero-mean functions, and Γ acts trivially on the former space $\mathbb{C}$ of constant functions. Hence

$$\bar{H}^n(\Gamma \times M, \operatorname{Ind}_\Lambda^{\Gamma\times M} 1_\Lambda) \cong \bar{H}^n(\Gamma \times M, 1_{\Gamma\times M}) \oplus \bar{H}^n(\Gamma \times M, L_0^2(X/\Lambda)).$$

By ergodicity the zero-mean regular representation cannot contain $1_{G\times M}$, so as $\Gamma \times M$ has property (H_T^n), it follows that $\bar{H}^n(\Gamma \times M, L_0^2(X/\Lambda)) = 0$. Thus

$$\bar{H}^n(\Lambda, 1_\Lambda) \hookrightarrow \bar{H}^n(\Gamma \times M, 1_{\Gamma\times M}) \cong \bar{H}^n(\Gamma, 1_{\Gamma\times M}^M) = \bar{H}^n(\Gamma, 1_\Gamma),$$

where the isomorphism follows from Lemma ??. This proves that $b_n(\Lambda) \leqslant b_n(\Gamma)$. $\square$

Here is another application of the preceding results and techniques.

Corollary 9 (Weak Nomizu Lemma). *If Γ is a finitely generated torsion-free nilpotent group with Mal'cev completion G, then*

$$H^n(\Gamma, 1) \cong H^n(G, 1),$$

where 1 is the relevant trivial representation.

Proof. As above, we have that $\operatorname{Ind}_\Gamma^G 1 = \mathbb{C} \oplus L_0^2(G/\Gamma)$. By Corollary 3, we have that $\bar{H}^n_{cts}(G, L_0^2(G/\Gamma)) = 0$. Thus $\bar{H}^n_{cts}(G, \operatorname{Ind}_\Gamma^G 1)$, and Corollary 5 and finite-dimensionality of either $H^n(\Gamma, 1)$ (as it is the cohomology of a finite-dimensional connected smooth manifold), or $H^n(G, 1)$ (as it is the Lie algebra cohomology of a f.d. nilpotent Lie algebra). □

We will now provide a proof of the main theorem that the Betti numbers are a quasi-isometry invariant among the finitely generated nilpotent groups.

Proof of Theorem 1. Analogous to before, we may assume by Lemma ?? that the finitely generated nilpotent groups Γ, Λ are torsion-free. As mentioned in the proof of Corollary 9, Γ and Λ have finite Betti numbers of all ranks; by Corollary 8 they have (H_T^n) for all $n \geqslant 1$, and are amenable. It follows from Theorem 6 that $b_n(\Gamma) = b_n(\Lambda)$ for all n. □

Here is an application. Let Γ_n be the following finitely generated, torsion-free nilpotent group:

$$\Gamma_n = \langle x_1, \ldots, x_n \mid [x_i, x_j] = x_{i+j}^{j-i} \text{ whenever } i + j \leqslant n \rangle.$$

Importantly, Γ_n embeds as a cocompact lattice inside the Lie group G_n associated to the nth *filiform Lie algebra* $\mathfrak{g}_n$, defined by the analogous commutation relations. Let $\mathfrak{g}_n^\infty$ be the associated graded algebra to $\mathfrak{g}_n$, which is also defined over $\mathbb{Q}$. In particular, the associated Lie group G_n^∞ admits a cocompact lattice Γ_n^∞.

Corollary 10. *If $n \geqslant 7$, the group Γ_n is not quasi-isometric to Γ_n^∞.*

Proof. Let $\mathfrak{g}_n^\infty$ be the associated graded Lie algebra to $\mathfrak{g}_n$, or the Lie algebra of the Lie group G_n^∞. It is calculated in [6] that

$$b_2(\mathfrak{g}_n) = 3, \qquad b_2(\mathfrak{g}_n^\infty) = \left\lfloor \frac{1}{2}(n+1) \right\rfloor,$$

so when $n \geqslant 7$ we have that $b_2(\mathfrak{g}_n) = b_2(\Gamma_n)$ (see e.g. Corollary 9) is not $b_2(\mathfrak{g}_n^\infty) = b_2(\Gamma_n^\infty)$. □

Corollary 11. *If $n \geqslant 7$, the group Γ_n is not quasi-isometric to G_n^∞ with any left-invariant sub-Finsler distance. In particular, Γ_n is not quasi-isometric to its asymptotic cone.*

Proof. By Chow's theorem (see Le Donne Theorem 4.1.8), any sub-Finsler metric induces the manifold topology on G_n^∞. In particular, Γ_n^∞ is cocompact and acts by isometries on G_n^∞, so Γ_n^∞ cannot be quasi-isometric to G_n^∞. □

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