Subsubnet convergence

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Recall that a *net* $a : I \to X$ is a mapping from a directed set $(I, \leqslant)$ to a space X. Here a preordered set is *directed* if for every $i, j \in I$ there is some k so that $k \geqslant i$ and $k \geqslant j$. A *tail* of a net is the set $\{j : j \geqslant i\}$ for some i in I. Note that the directed condition on I means that every two tails meet in a tail.

A net is said to *converge* to a point $x \in X$ if for every neighborhood U of x, there is a tail (and hence eventually all tails) T so that for all $t \in T$, $a_t \in U$.

A subset I' of I is said to be *cofinal* if every element of I is dominated by an element of I'. Given a net $a : I \to X$, a *subnet* $f^*a : J \to I \to X$ is the pullback of a by a monotone-increasing map f with cofinal image. Note that if a net converges to $x \in X$, then these conditions guarantee that all subnets converge to x.

This note is based around the following lemma, which is much more useful than one would expect in verifying convergence. I personally like this lemma because using it feels like "black magic."

Lemma 1. *Suppose $a : I \to X$ is a net in the topological space X, and $x \in X$. If every subnet f^*a has a subnet $(fg)^*a$ which converges to x, then a converges to x.*

Proof. We prove the contrapositive by supposing non-convergence, and finding a subnet whose every subnet does not converge.

So suppose that a does not converge to x, and let U be a neighborhood of x. Since a does not converge to x, the subset of elements of I for which $a_i \notin U$ is cofinal in I, so if ι is the inclusion of this subset, we have that the subset ι^*a is disjoint from U. But then ι^*a cannot have any subnet which converges to x, as claimed. $\square$

This lemma has various nice applications, but the main technique is the following: if you know what a limit *should be*, then by applying a compactness argument to every subsequence you get convergence of the original sequence. This is illustrated in the following example.

The *weak topology* on a vector space X is the weakest topology which makes the linear functionals $X^* = \{X \to \mathbb{R}\}$ continuous, and a linear mapping between normed vector spaces is said to be *bounded* if it is continuous in norm, and is *compact* if the image of the unit ball in the domain is precompact in the codomain. Note that compact operators are always bounded.

Proposition 1. *Compact operators upgrade weak convergence to norm convergence. That is, if $x_n \to x$ is a weakly convergent net and $L : X \to Y$ is compact, then $Lx_n \to Lx$ in norm.*

Proof. Suppose that $x_n \to x$ is weakly convergent. Weakly convergent sequences are norm bounded, and bounded operators are weakly continuous, so $\{Lx_n\}$ is contained in a (norm) compact subset of Y, and is weakly convergent.

If we were to apply compactness, we would just get a subnet of $\{Lx_n\}$ which converges to Lx, since norm convergence implies weak convergence and the weak topology is Hausdorff. This does not imply that $Lx_n \to Lx$ by itself.

To get around this, we use "black magic." That is, we let $\{f^* Lx_n\}_n$ be any generic subnet of $\{Lx_n\}$. This subnet still weakly converges to Lx, and lies in a compact set, so it has a norm-convergent subnet which converges to Lx. By the lemma, we are done: $Lx_n \to Lx$ in norm. $\square$

Weird!